Istituzioni di Matematiche
CdL Scienze Biologiche

Applicazioni al 2° teorema di sostituzioni

(2) $\int R(\operatorname{sen}^2 x, \cos^2 x)\, dx$

$$\boxed{\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x}}$$

$$\frac{1}{1+\operatorname{tg}^2 x} = \frac{1}{1+\frac{\operatorname{sen}^2 x}{\cos^2 x}} = \frac{1}{\frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x}} = \frac{\cos^2 x}{\cos^2 x + \operatorname{sen}^2 x}$$

$$= \cos^2 x$$

Analogamente

$$\frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{\frac{\operatorname{sen}^2 x}{\cos^2 x}}{1+\frac{\operatorname{sen}^2 x}{\cos^2 x}} = \frac{\frac{\operatorname{sen}^2 x}{\cancel{\cos^2 x}}}{\frac{\cos^2 x + \operatorname{sen}^2 x}{\cancel{\cos^2 x}}} = \operatorname{sen}^2 x$$

Da cui

$$\int R(\operatorname{sen}^2 x, \cos^2 x)\, dx = \int R\left(\frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}, \frac{1}{1+\operatorname{tg}^2 x}\right) dx$$

Suggerimento: usa il II° sostituzione con

$$f(t) = \operatorname{arctg} t \quad , \quad f'(t) = \frac{1}{1+t^2}$$

$$\Longrightarrow \int R(\sin^2 x, \cos^2 x)\, dx = \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}\right) \cdot \frac{1}{1+t^2}\, dt$$

Esempio $\displaystyle \int \frac{\cos^2 x}{1 - 2\sin^2 x}\, dx =$

$$= \int \frac{\frac{1}{1+\operatorname{tg}^2 x}}{1 - 2\frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}}\, dx = \quad t = \operatorname{tg} x \Leftrightarrow \underbrace{x = \operatorname{arctg} t}_{\text{suggerim.}}$$

$f(t) = \operatorname{arctg} t$

$f'(t) = \frac{1}{1+t^2}$

2° sost $\displaystyle \int \frac{\frac{1}{1+t^2}}{1 - 2\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\, dt =$

$$= \int \frac{\frac{1}{(1+t^2)^{\cancel{2}}}}{\frac{1+t^2-2t^2}{\cancel{(1+t^2)}}}\, dt = \int \frac{1}{(1+t^2)(1-t^2)}\, dt$$

Passo 2 Decomp. denominatore $1+t^2$ è irriduc.

$$1-t^2 = (1-t)(1+t)$$

ossia

$$\frac{1}{(1+t^2)(1-t^2)} = \frac{1}{(1-t)(1+t)(1+t^2)}$$

Passo 3 Ricerca costanti

$$\boxed{\frac{1}{(1-t)(1+t)(1+t^2)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{Ct+D}{1+t^2}}$$

$$= \frac{A(1+t)(1+t^2) + B(1-t)(1+t^2) + (Ct+D)(1-t^2)}{(1-t)(1+t)(1+t^2)}$$

$$= \frac{At^3 + At^2 + At + A - Bt^3 + Bt^2 - Bt + B - Ct^3 - Dt^2 + Ct + D}{(1-t)(1+t)(1+t^2)}$$

$$= \frac{(A-B-C)t^3 + (A+B-D)t^2 + (A-B+C)t + (A+B+D)}{(1-t)(1+t)(1+t^2)}$$

$$\begin{cases} A-B-C=0 \\ A+B-D=0 \\ A-B+C=0 \\ A+B+D=1 \end{cases}$$

$A-B-C=0 \rightarrow C = A-B$

$C=0$

$A-B+C=0 \rightarrow A-B+\underbrace{A-B}_{C}=0$

$\Rightarrow 2A-2B=0$

$\Rightarrow A=B$

4ª

$$\Rightarrow A = B$$

$$D = A + B = 2A$$

$$\xrightarrow{4^\circ} A + A + 2A = 1 \Rightarrow A = \frac{1}{4}, \quad B = \frac{1}{4}, \quad C = 0, \quad D = \frac{1}{2}$$

Passo 4 Sostituisco i valori delle costanti dal Passo 3

$$\int \frac{1}{(1-t)(1+t)(1+t^2)}\,dt = \frac{1}{4}\int \frac{1}{1-t}\,dt + \frac{1}{4}\int \frac{1}{1+t}\,dt + \frac{1}{2}\int \frac{1}{t^2+1}\,dt$$

$$= -\frac{1}{4}\lg|1-t| + \frac{1}{4}\lg|1+t| + \frac{1}{2}\operatorname{arctg} t =$$

$t = \operatorname{tg} x$

$$= -\frac{1}{4}\lg|1-\operatorname{tg} x| + \frac{1}{4}|1+\operatorname{tg} x| + \frac{1}{2}x + \text{cost}$$

A parte:

$$\int \frac{1}{1-t}\,dt = \qquad z = 1-t \rightarrow dz = -dt$$

$$\int \frac{\ \ }{1-t}\, dt =$$

$$= - \int \frac{-dt}{1-t} \overset{1^a \text{ sost}}{=} - \int \frac{dz}{z}$$

$$= - \lg|z| = - \lg|1-t|$$

Applicazione (3)

$$\int R\left(x, \sqrt{\frac{ax+b}{cx+d}}\right) dx$$

In questi casi usa 2^a sostituzione con:

$$h(x) = \sqrt{\frac{ax+b}{cx+d}} = t$$

Si prova che $h(x)$ è invertibile. Allora scegliere

$$F(t) = h^{-1}(t)$$

Esempio: $\int \frac{1}{x} \cdot \sqrt{\frac{3x-2}{3x+1}}\, dx$

$$t = \sqrt{\frac{3x-2}{3x+1}} \iff t^2 = \frac{3x-2}{3x+1}$$

$$3x\,t^2 + t^2 = 3x - 2$$

$$3x\,t^2 - 3x = -2 - t^2$$

$$x\,(3t^2 - 3) = -2 - t^2$$

$$F(t) = \boxed{x = \frac{-2 - t^2}{3t^2 - 3}}$$

$$F'(t) = \frac{-2\cancel{t}\,(\cancel{3t^2}-3) - (-2-\cancel{t^2})\cdot \cancel{6t}}{(3t^2-3)^2} =$$

$$= \frac{6t + 12t}{(3t^2-3)^2} = \frac{18\,t}{(3t^2-3)^2}$$

$$= \frac{\overset{2}{\cancel{18}}\,t}{\cancel{3^2}\,(t^2-1)^2} = \frac{2t}{(t^2-1)^2}$$

Sostituendo

$$\int \frac{1}{x}\sqrt{\frac{3x-2}{3x+1}}\,dx = \int \underbrace{\frac{1}{\frac{-2-t^2}{3t^2-3}}}_{x} \cdot t \cdot \underbrace{\frac{2t}{(t^2-1)^2}}_{F'(t)}\,dt =$$

$$= \int \frac{3\,\cancel{(t^2-1)}}{-2-t^2}\,\frac{2t^2}{(t^2-1)^{\cancel{2}}} = -6\int \frac{t^2}{(2+t^2)\,(t-1)\,(t+1)}$$

Passo 3 Ricavo costanti

$$\boxed{\frac{t^2}{(2+t^2)(t-1)(t+1)} = \frac{At+B}{2+t^2} + \frac{C}{t-1} + \frac{D}{t+1}}$$

$$= \frac{(At+B)(t^2-1)+C(t+1)(2+t^2)+D(t-1)(2+t^2)}{(2+t^2)(t-1)(t+1)}$$

$$= \frac{At^3+Bt^2-At-B+Ct^3+Ct^2+2Ct+2C+Dt^3-Dt^2+2Dt-2D}{(2+t^2)(t-1)(t+1)}$$

$$= \frac{(A+C+D)t^3+(B+C-D)t^2+(-A+2C+2D)t+(-B+2C-2D)}{(2+t^2)(t-1)(t+1)}$$

Da cui

$$\begin{cases} A+C+D=0 \\ B+C-D=1 \\ -A+2C+2D=0 \\ -B+2C-2D=0 \end{cases}$$

$A+C+D=0 \longrightarrow D=-A-C$

$3^e \longrightarrow -A+\cancel{2C}+2(\underbrace{-A-\cancel{C}}_{D})=0$

$\Rightarrow -3A=0 \qquad A=0$

$D=-C$

$$2^e \longrightarrow B+C-(\underbrace{-C}_{D})=1 \Rightarrow B+2C=1$$

$$B=1-2C$$

$$4^e \longrightarrow -(1-2C)+2C-2(-C)=0$$

$$4^e \Longrightarrow \quad -\underbrace{(1-2C)}_{B} + 2C - 2\underbrace{(-C)}_{D} = 0$$

$$\Longrightarrow -1 + 2C + 2C + 2C = 0 \Longrightarrow 6C = +1$$

$$B = 1 - 2 \cdot \left(-\frac{1}{6}\right) = 1 + \frac{1}{3} = \frac{4}{3}$$

$$C = +\frac{1}{6}$$

$$D = -C = -\frac{1}{6}$$

Da cui

$$A = 0 \qquad B = \frac{2}{3} \qquad C = +\frac{1}{6} \qquad D = -\frac{1}{6}$$

Passo sostituisco all'espressione del Passo 3

$$\int \frac{t^2}{(2+t^2)(t-1)(t+1)}\,dt = \frac{2}{3}\int \frac{1}{2+t^2}\,dt + \frac{1}{6}\int \frac{1}{t-1}\,dt - \frac{1}{6}\int \frac{1}{t+1}$$

$$= \frac{\cancel{2}}{3}\,\frac{1}{\cancel{2}}\int \frac{1}{\frac{t^2}{2}+1}\,dt + \frac{1}{6}\lg|t-1| - \frac{1}{6}\lg|t+1|$$

$$= \frac{1}{3}\int \frac{1 \cdot dt}{\left(\frac{t}{\sqrt{2}}\right)^2+1} + \frac{1}{6}\lg|t-1| - \frac{1}{6}\lg|t+1|$$

$$= \frac{\sqrt{2}}{3}\int \frac{\frac{1}{\sqrt{2}}\,dt}{\left(\frac{t}{\sqrt{2}}\right)^2+1} + \rule{4cm}{0.4pt}$$

$$\overset{1^\circ \text{ sost}}{=} \frac{\sqrt{2}}{3} \left[\int \frac{dz}{z^2+1} \right]_{z=\frac{t}{\sqrt{2}}} + \frac{1}{6} \lg|t-1| - \frac{1}{6} \lg|t+1|$$

$$= \frac{\sqrt{2}}{3} \operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{6} \lg|t-1| - \frac{1}{6} \lg|t+1|$$

$$\overset{\left(t=\sqrt{\frac{3x-2}{3x+1}}\right)}{=} \frac{\sqrt{2}}{3} \operatorname{arctg}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{3x-2}{3x+1}}\right) + \frac{1}{6} \lg\left|\sqrt{\frac{3x-2}{3x+1}} - 1\right| - \frac{1}{6} \lg\left|\sqrt{\frac{3x-2}{3x+1}} + 1\right|$$

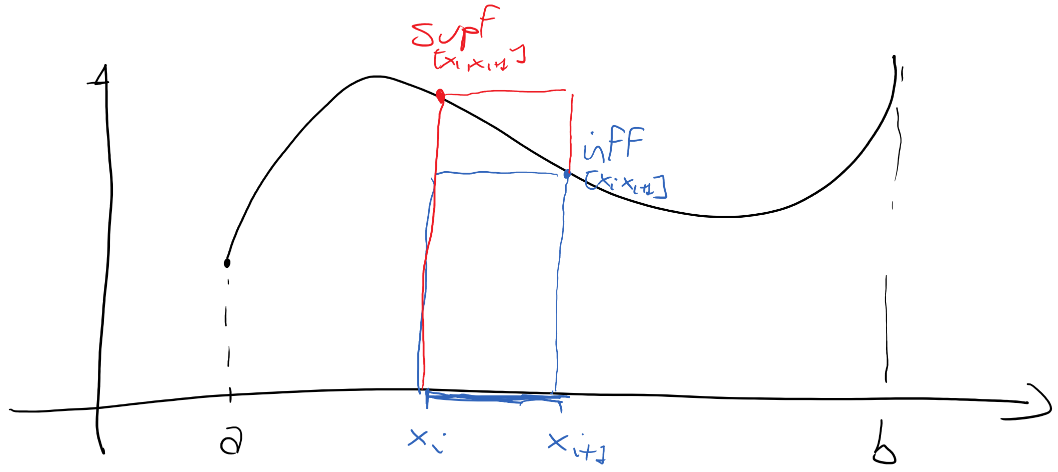
Integrale di Riemann

Sia $f : [a,b] \to \mathbb{R}$ limitata.

In $[a,b]$ scelgo un numero finito di punti

$$x_0 = a < x_1 < x_2 < \dots < x_n = b$$

Chiameremo $D = \{x_0, x_1, \dots, x_n\}$ decomposizione di $[a,b]$



Considero

$\inf \quad \sum^{n-1} \inf f(x) \, (x \dots$

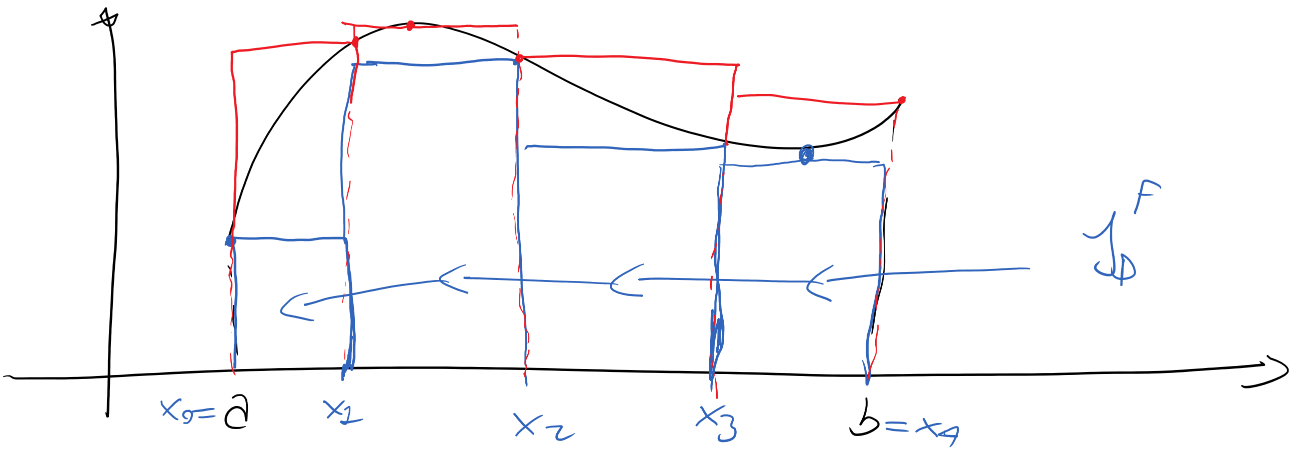
Considero

$$\mathcal{J}_D^f = \sum_{i=0}^{n-1} \inf_{[x_i, x_{i+1}]} f(x) \, (x_{i+1} - x_i)$$

$$S_D^f = \sum_{i=0}^{n-1} \sup_{[x_i, x_{i+1}]} f(x) \, (x_{i+1} - x_i)$$

$\mathcal{J}_D^f$ viene detto somma inferiore relativo a D

S_D^f viene detta somma superiore relativa alla decomp. D



Definiamo

$$\sigma = \{ \mathcal{J}_D^f : D \text{ decomposiz. di } [a,b] \} \subseteq \mathbb{R}$$

$$\Sigma = \{ S_D^f : D \text{ decomp. di } [a,b] \} \subseteq \mathbb{R}$$

Nota: Fissato D dec. di $[a,b]$

$$\mathcal{J}_D^f = \sum^{n-1} \inf f(x) \, (x_{i+1} - x_i)$$

$$s_D^f = \sum_{i=0}^{n-1} \inf_{[x_i, x_{i+1}]} f(x)\,(x_{i+1}-x_i)$$

$$\le \sum_{i=0}^{n-1} \sup_{[x_i, x_{i+1}]} f(x)\,(x_{i+1}-x_i) = S_D^f$$

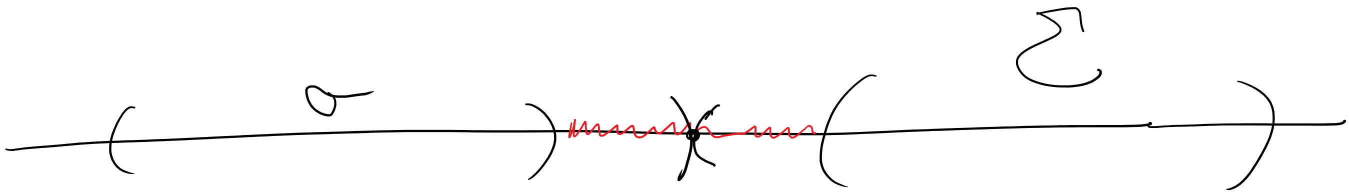
<u>Lemma</u> (Riemann)

$\forall\, D_1, D_2$ decomp. di $[a,b]$

$$\Longrightarrow \quad s_{D_1}^f \le S_{D_2}^f$$

Ciò significa

$$\sigma \le \Sigma$$



<u>Def</u> (Funzione Riemann integrabile)

Sia $f : [a,b] \to \mathbb{R}$ limitata. $f(x)$ si dice Riemann-integrabile se

$$\sup \sigma = \inf \Sigma = c \in \mathbb{R}$$

e in tal caso definiamo

$$\int_a^b f(x)\,dx = c$$

ossia l'integrale (che è un numero!) coincide con l'unico punto di contiguità tra σ e Σ

In particolare

$$\int_a^b f(x)\,dx = \text{area sottesa al grafico di } f(x)$$

Nota:

$$s_D^f \leq \int_a^b f(x)\,dx \leq S_D^f$$

per ogni decomp D di $[a,b]$

Se scelgo come $D = \{a, b\}$

$$\inf_{[a,b]} f(x) \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \sup_{[a,b]} f(x)\,(b-a)$$

Dividendo per $(b-a)$ si ottiene

Teorema (Media Integrale)

Se $f : [a,b] \to \mathbb{R}$ è Riemann integrabile

Allora

$$\boxed{\inf_{[a,b]} f(x) \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \sup_{[a,b]} f(x)}$$

Se, in particolare, $f : [a,b] \to \mathbb{R}$ è continua, dal teorema di Darboux otteniamo

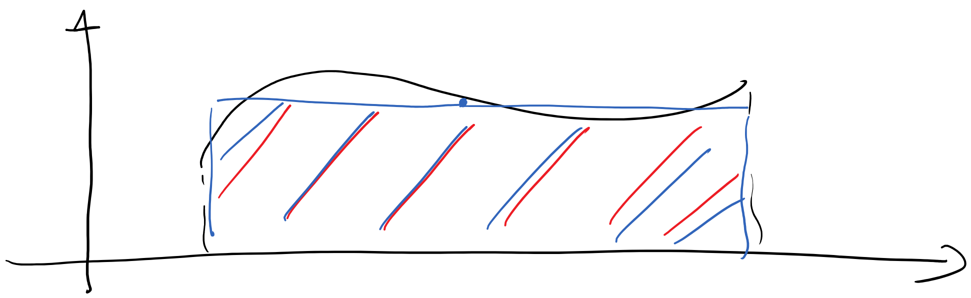
Corollario

$f : [a,b] \to \mathbb{R}$ è continua

$\Longrightarrow \quad \exists\, c \in [a,b]$

$$\int_a^b f(x)\, dx = f(c)\,(b-a)$$

ossia, l'area sottesa al grafico della funzione si può sempre pensare come area di un rettangolo



Esercizio

$$\int \frac{3x-1}{x^2-5x+6}\, dx$$

Passo 2

Decomp. denom.

$$\Delta = 25 - 24 = 1 \quad \to \quad x_{1,2} = \frac{5 \pm 1}{2} \begin{cases} 2 \\ 3 \end{cases}$$

$$\Longrightarrow \quad x^2 - 5x + 6 = (x-2)\cdot(x-3)$$

$$\Rightarrow \quad x^2-5x+6 = (x-2)\cdot(x-3)$$

Passo 3 Ricerca costanti

$$\boxed{\frac{3x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}}$$

$$= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}$$

$$\begin{cases} A+B=3 \longrightarrow B = 3-A \\ -3A-2B=-1 \end{cases}$$

$$\longrightarrow -3A-2\underbrace{(3-A)}_{B} = -1$$

$$-3A-6+2A = -1$$

$$-A = 5 \Rightarrow A = -5$$

$$B = 8$$

Passo 4 Sostituisco le costanti al Passo 3

$$\int \frac{3x-1}{x^2-5x+6}\,dx = -5\int \frac{1}{(x-2)}\,dx + 8\int \frac{1}{x-3}\,dx$$

$$= -5 \lg|x-2| + 8 \lg|x-3| + \text{cost}$$

Esercizio

$$\begin{cases} a_1 = \frac{1}{2} \\ a_{n+1} = a_n^2 \end{cases}$$

$a_1 = \frac{1}{2} > 0$, $a_2 = \left(\frac{1}{2}\right)^2 > 0$ ... $a_{n+1} = a_n^2 > 0$ $\forall n \in \mathbb{N}$

$(a_n)_{n\in\mathbb{N}}$ è formato da numeri positivi

$$\boxed{f(x) = x^2}$$

$a_{n+1} = f(a_n)$

$f'(x) = 2x > 0$ $\forall x \in]0, +\infty[$

Sto studiando la monotonia di $f(x)$ su $]0, +\infty[$

poiché $a_n \in]0, +\infty[$ $\forall n \in \mathbb{N}$

$\Longrightarrow$ $f(x)$ cresce su $]0, +\infty[$

$a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_2 < a_1$

$a_3 = f(a_2) < f(a_1) = a_2$

$a_4 < a_3$ ... ossia $(a_n)_{n\in\mathbb{N}}$ è decrescente

$$\boxed{a_{n+1} < a_n \quad \forall n \in \mathbb{N}}$$

Poiché $a_n > 0$ $\forall n \in \mathbb{N}$ (limitata inferiorm.)

$\Longrightarrow$ $\exists \lim a_n = l = \inf\{a_n : n \in \mathbb{N}\}$ CD

$$\Longrightarrow \exists \lim_{n\to+\infty} a_n = l = \inf\{a_n : n\in\mathbb{N}\} \in \mathbb{R}$$

$$l \geq 0$$

$F(x)$ è continua $\Longrightarrow$ $l = F(l)$

$$l = l^2$$

$$l^2 - l = 0 \quad \begin{cases} l = 0 \\ l = 1 \end{cases}$$

$$l \leq a_n \quad \forall n\in\mathbb{N} \Longrightarrow l \leq a_1 = \frac{1}{2}$$

$$\Longrightarrow l = 0$$

$$\Longrightarrow \boxed{\lim_{n\to+\infty} a_n = 0}$$

Esercizio

$$\begin{cases} a_1 = 3 \\ a_{n+1} = 2(a_n - 1) \end{cases}$$

$$\boxed{F(x) = 2(x-1)} \qquad \boxed{a_{n+1} = F(a_n)}$$

$$\boxed{F(x) = 2(x-1)} \qquad \boxed{a_{n+1} = F(a_n)}$$

$$F'(x) = 2 > 0 \Rightarrow F(x) \text{ cresce}$$

$$a_1 = 3 \qquad a_2 = 2(3-1) = 4$$

$$a_1 < a_2 \Longrightarrow a_n < a_{n+1} \quad \forall n \in \mathbb{N}$$

$\Rightarrow (a_n)_{n\in\mathbb{N}}$ è crescente

$$\Rightarrow \lim_{n\to+\infty} a_n = l = \sup\{a_n : n\in\mathbb{N}\} \begin{cases} \nearrow +\infty \\ \searrow \in \mathbb{R} \end{cases}$$

Supponiamo che $l \in \mathbb{R}$

$$\Rightarrow \quad l = F(l)$$

$$l = 2(l-1)$$

$$l - 2l + 2 = 0 \Rightarrow -l = -2$$

$$\Rightarrow l = 2$$

Domanda $\quad a_n \leq 2 \quad \forall n\in\mathbb{N}$?

ma $\quad a_1 = 3 < 2$ No

$$\Longrightarrow \lim_{n\to+\infty} a_n = +\infty$$

Esercizio

$$\int \frac{x-1}{x^3-1}\,dx$$

Passo 2 Dec. denominatore

$$P(x) = x^3 - 1 = ?$$

radici ± 1

$P(-1) = (-1)^3 - 1 = -2$ No

$P(1) = 1^3 - 1 = 0$ ok

Uso Ruffini

$$\begin{array}{c|ccc|c}
 & 1 & 0 & 0 & -1 \\
1 & & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & \#
\end{array}$$

$$\underbrace{1 \quad 1 \quad 1}_{x^2+x+1}$$

$$\Longrightarrow x^3 - 1 = (x-1)(x^2+x+1)$$

radice

irriducibile

$\Delta = 1 - 4 < 0$

$$\int \frac{x-1}{x^3-1}\,dx = \int \frac{\cancel{x-1}}{\cancel{(x-1)}(x^2+x+1)}\,dx = \int \frac{1}{x^2+x+1}\,dx = (*)$$

$$(x^2+x)+1 = \left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\left(x+\frac{1}{2}\right)^2$$

$$(*) = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \frac{1}{\frac{3}{4}}\int \frac{1}{\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1}\,dx$$

$$= \frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\,dx =$$

$$= \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}\,dx$$

$$z = \frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}} \longrightarrow dz = \frac{2}{\sqrt{3}}\,dx$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{3}_{\sqrt{3}}}\,\frac{\cancel{\sqrt{3}}}{\cancel{2}}\int \frac{\frac{2}{\sqrt{3}}\,dx}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1} \overset{1^e\ \text{sost}}{=}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dz}{z^2+1} = \frac{2}{\sqrt{3}} \operatorname{arctg} z$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} \left(\frac{2}{\sqrt{3}} x + \frac{1}{\sqrt{3}} \right) + \underline{\underline{\text{cost}}}$$